QUASI-POISSON ACTIONS AND MASSIVE NON-ROTATING BTZ BLACK HOLES

SÉBASTIEN RACANIÈRE

Abstract. Using ideas from an article of P. Bieliavsky, M. Rooman and Ph. Spindel on BTZ black holes, I construct a family of interesting examples of quasi-Poisson actions as defined by A. Alekseev and Y. Kosmann-Schwarzbach. As an application, I obtain a genuine Poisson structure on $\mathrm{SL}(2,\mathbb{R})$ which induces a Poisson structure on a BTZ black hole.

Acknowledgement

I would like to thank Pierre Bieliavsky for advising me to look at the article [4] and then answering my questions about it. Thank you to Stéphane Detournay for talking to me about physics in a way I could understand and for answering many questions I had about [3]. And finally, a big thanks to David Iglesias-Ponte for interesting conversations about [1].

The author was supported by a Marie Curie Fellowship, EC Contract Ref: HPMF-CT-2002-01850

1. Introduction

In [4], P. Bieliavsky, M. Rooman and Ph. Spindel construct a Poisson structure on massive non-rotating BTZ black holes; in [3], P. Bieliavsky, S. Detournay, Ph. Spindel and M. Rooman construct a star product on the same black hole. The direction of this deformation is a Poisson bivector field which has the same symplectic leaves as the Poisson bivector field of [4]: roughly speaking, they correspond to orbits under a certain twisted action by conjugation.

In the present paper, I wish to show how techniques used in [4] in conjunction with techniques of the theory of quasi-Poisson manifolds (see [1] and [2]) can be used to construct an interesting family of manifolds with a quasi-Poisson action and how a particular case of this family leads to a genuine Poisson structure on a massive non-rotating BTZ black hole with similar symplectic leaves as in [4] and [3].

2. Main results

I will not recall here the basic definitions in the theory of quasi-Poisson manifolds and quasi-Poisson actions. The reader will find these definitions in A. Alekseev and Y. Kosmann-Schwarzbach [1], and in A. Alekseev, Y. Kosmann-Schwarzbach and E. Meinrenken [2].

Let G be a connected Lie group of dimension n and $\mathfrak{g}$ its Lie algebra, on which G acts by the adjoint action Ad. Assume we are given an Ad-invariant non-degenerate bilinear form K on $\mathfrak{g}$. For example, if G is semi-simple, then K could be the Killing form. In the following, I will denote by K again the linear isomorphism

$$\begin{array}{ccc} \mathfrak{g} & \longrightarrow & \mathfrak{g}^* \\ x & \longmapsto & K(x,\cdot). \end{array}$$

Let $D = G \times G$ and $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$ its Lie algebra. Define an Ad-invariant non-degenerate bilinear form $\langle\, , \rangle$ of signature (n, n) by

$$\begin{array}{ccc} \mathfrak{d} \times \mathfrak{d} = (\mathfrak{g} \oplus \mathfrak{g}) \times (\mathfrak{g} \oplus \mathfrak{g}) & \longrightarrow & \mathbb{R} \\ ((x, y), (x', y')) & \longmapsto & K(x, x') - K(y, y'). \end{array}$$

Assume there is an involution σ on G which induces an orthogonal involutive morphism, again denoted by σ, on $\mathfrak{g}$. Let $\Delta_+ : G \to D$ and $\Delta_+^\sigma : G \to D$ be given by

$$\Delta_+(g) = (g, g)$$

and

$$\Delta_+^\sigma(g) = (g, \sigma(g)).$$

Denote by G_+ and G_+^σ their respective images in D. Let $S = D/G_+$ and $S^\sigma = D/G_+^\sigma$. Then both S and S^σ are isomorphic to G. The isomorphism between S and G is induced by the map

$$\begin{array}{ccc} D & \longrightarrow & G \\ (g, h) & \longmapsto & gh^{-1}, \end{array}$$

whereas the isomorphism between S^σ and G is induced by

$$\begin{array}{ccc} D & \longrightarrow & G \\ (g, h) & \longmapsto & g\sigma(h)^{-1}. \end{array}$$

I will use these two isomorphisms to identify S and G, and S^σ and G. Denote again by $\Delta_+ : \mathfrak{g} \to \mathfrak{d}$ and $\Delta_+^\sigma : \mathfrak{g} \to \mathfrak{d}$ the morphisms induced by $\Delta_+ : G \to D$ and $\Delta_+^\sigma : G \to D$ respectively. Let $\Delta_- : \mathfrak{g} \to \mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$ and $\Delta_-^\sigma : \mathfrak{g} \to \mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$ be defined by

$$\Delta_-(x) = (x, -x),$$

and

$$\Delta_-^\sigma(x) = (x, -\sigma(x)).$$

Let $\mathfrak{g}_- = \mathrm{Im}(\Delta_-)$ and $\mathfrak{g}_-^\sigma = \mathrm{Im}(\Delta_-^\sigma)$. We have two quasi-triples $(D, G_+, \mathfrak{g}_-)$ and $(D, G_+^\sigma, \mathfrak{g}_-^\sigma)$. They induce two structures of quasi-Poisson Lie group on D, of respective bivector fields P_D and P_D^σ, and two structures of quasi-Poisson Lie group on G_+ and G_+^σ of respective bivector fields P_{G_+} and $P_{G_+^\sigma}$. I will simply write G_+, respectively G_+^σ, to denote the group together with its quasi-Poisson structure. Of course, these quasi-Poisson structures are pairwise isomorphic. More precisely, the isomorphism $\mathrm{Id} \times \sigma : (g, h) \longmapsto (g, \sigma(h))$ of D sends P_D on P_D^σ and vice-versa. This isomorphism can be used to deduce some of the results given at the beginning of the present article from the results of Alekseev and Kosmann-Schwarzbach [1]; but it takes just as long to redo the computations, and that is what I do here.

According to [1], the bivector field P_D, respectively P_D^σ, is projectable onto S, respectively S^σ. Let P_S and $P_{S^\sigma}^\sigma$ be their respective projections. Using the identifications between S and G, and S^σ and G, one can check that P_S and $P_{S^\sigma}^\sigma$ are the same bivector fields on G. What is more interesting, and what I will prove, is the following Theorem.

Theorem 2.1. *The bivector field P_D^σ is projectable onto S. Let P_S^σ be its projection. Identify S with G and trivialise their tangent space using right translations, then for s in S and ξ in $\mathfrak{g}^* \simeq T_s^*S$ there is the following explicit formula*

$$P_S^\sigma(s)(\xi) = \frac{1}{2}(Ad_{\sigma(s)^{-1}} - Ad_s) \circ \sigma \circ K^{-1}(\xi).$$

Moreover, the action

$$\begin{array}{ccc} G_+^\sigma \times S & \longrightarrow & S \\ (g, s) & \longmapsto & gs\sigma(g)^{-1} \end{array}$$

of G_+^σ *on* (S, P_S^σ) *is quasi-Poisson in the sense of Alekseev and Kosmann-Schwarzbach* [1]*. The image of* $P_S^\sigma(s)$*, seen as a map* $T_s^*S \longrightarrow T_sS$*, is tangent to the orbit through* s *of the action of* G^σ *on* S*.*

In the setting of the above Theorem, the bivector field P_S^σ is G^σ invariant; hence if F is a subgroup of G^σ and $\mathbf{I}$ is an F-invariant open subset of S such that the action of F on $\mathbf{I}$ is principal then $F\backslash\mathbf{I}$ is a smooth manifold and P_S^σ descends to a bivector field on it. An application of this remark is the following Theorem.

Theorem 2.2. *Let* $G = \mathrm{SL}(2,\mathbb{R})$*. Let*

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and choose $\sigma = \mathrm{Ad}_H$*. Let*

$$\mathbf{I} = \{ \begin{bmatrix} u+x & y+t \\ y-t & u-x \end{bmatrix} \mid \mathbf{u^2 - x^2 - y^2 + t^2 = 1, t^2 - y^2 > 0} \}$$

be an open subset of S*. Let* F *be the following subgroup of* G

$$F = \{\exp(n\pi H), n \in \mathbb{N}\}.$$

The quotient $F\backslash\mathbf{I}$ *(together with an appropriate metric) is a model of massive non-rotating BTZ black hole (see* [4]*). The bivector field it inherits following the above remark, is Poisson. Its symplectic leaves consist of the projection to* $F\backslash\mathbf{I}$ *of the orbits of the action of* G^σ *on* S *except along the projection of the orbit of the identity. Along this orbit, the bivector field vanishes and each point forms a symplectic leaf.*

In the coordinates (46) *of* [4] *(or (3) of the present article), the Poisson bivector field is*

$$2\cosh^2(\frac{\rho}{2})\sin(\tau)\sinh(\rho)\partial_\tau \wedge \partial_\theta. \tag{1}$$

The above Poisson bivector field should be compared with the one defined in [4] and given by

$$\frac{1}{\cosh^2(\frac{\rho}{2})\sin(\tau)}\partial_\tau \wedge \partial_\theta.$$

The symplectic leaves of this Poisson structure are the images under the projection $\mathbf{I} \longrightarrow \mathbf{F}\backslash\mathbf{I}$ of the action of G_+^σ on S.

3. Let the computations begin

Throughout the present article, I will use the notations introduced in the previous Section. To begin with, I will prove that $(D, G_+^\sigma, \mathfrak{g}_-^\sigma)$ does indeed form a quasi-triple.

Because $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$, one also has a decomposition $\mathfrak{d}^* = \mathfrak{g}^* \oplus \mathfrak{g}^*$. One also has $\mathfrak{d} = \mathfrak{g}_+^\sigma \oplus \mathfrak{g}_-^\sigma$ and accordingly $\mathfrak{d}^* = {\mathfrak{g}_+^\sigma}^* \oplus {\mathfrak{g}_-^\sigma}^*$. Denote $p_{\mathfrak{g}_+^\sigma}$ and $p_{\mathfrak{g}_-^\sigma}$ the projections on respectively $\mathfrak{g}_+^\sigma$ and $\mathfrak{g}_-^\sigma$ induced by the decomposition $\mathfrak{d} = \mathfrak{g}_+^\sigma \oplus \mathfrak{g}_-^\sigma$. So that $1_\mathfrak{d} = p_{\mathfrak{g}_+^\sigma} + p_{\mathfrak{g}_-^\sigma}$.
In this article, I express results using mostly the decomposition $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$. Using it, we have

$${\mathfrak{g}_+^\sigma}^* = \{(\xi, \xi \circ \sigma) \mid \xi \in \mathfrak{g}^*\}$$

and

$${\mathfrak{g}_-^\sigma}^* = \{(\xi, -\xi \circ \sigma) \mid \xi \in \mathfrak{g}^*\}.$$

Proposition 3.1. *The triple* $(D, G_+^\sigma, \mathfrak{g}_-^\sigma)$ *forms a quasi-triple in the sense of* [1]. *The characteristic elements of this quasi-triple as defined in* [1] *and hereby denoted by* j, $\mathbf{F}^\sigma$, φ^σ *and the r-matrix* $r_{\mathfrak{d}}^\sigma$ *are*

$$\begin{array}{rcc} j : \mathfrak{g}_+^{\sigma\,*} & \longrightarrow & \mathfrak{g}_-^\sigma \\ (\xi, \xi \circ \sigma) & \longmapsto & \Delta_-^\sigma \circ K^{-1}(\xi), \end{array}$$

and

$$F^\sigma = 0,$$

and

$$\begin{array}{rcl} \varphi^\sigma : \bigwedge^3 \mathfrak{g}_+^{\sigma\,*} & \longrightarrow & \mathbb{R} \\ ((\xi, \sigma \circ \xi), (\eta, \sigma \circ \eta), (\nu, \sigma \circ \eta)) & \longmapsto & 2K(K^{-1}(\nu), [K^{-1}(\xi), K^{-1}(\eta)]), \end{array}$$

and finally the r-matrix

$$\begin{array}{rcc} r_{\mathfrak{d}}^\sigma : \mathfrak{g}^* \oplus \mathfrak{g}^* & \longrightarrow & \mathfrak{g} \oplus \mathfrak{g} \\ (\xi, \eta) & \longmapsto & \frac{1}{2}\Delta_-^\sigma \circ K^{-1}(\xi + \eta \circ \sigma). \end{array}$$

Proof. It is straightforward to prove that $\mathfrak{d} = \mathfrak{g}_+^\sigma \oplus \mathfrak{g}_-^\sigma$ and that both $\mathfrak{g}_+^\sigma$ and $\mathfrak{g}_-^\sigma$ are isotropic in $(\mathfrak{d}, \langle\,\rangle)$. This proves that $(D, G^\sigma, \mathfrak{g}_-^\sigma)$ is a quasi-triple.

For $(\xi, \xi \circ \sigma)$ in $\mathfrak{g}_+^{\sigma\,*}$ and $(x, \sigma(x))$ in $\mathfrak{g}_+^\sigma$

$$\langle j(\xi, \xi \circ \sigma), (x, \sigma(x))\rangle = (\xi, \xi \circ \sigma)(x, \sigma(x)).$$

The map j is actually characterised by this last property. The equality

$$\langle \Delta_-^\sigma \circ K^{-1} \circ \Delta_+^{\sigma*}(\xi, \xi \circ \sigma), (x, \sigma(x)\rangle = (\xi, \xi \circ \sigma)(x, \sigma(x)),$$

proves that

$$\begin{array}{rcl} j(\xi, \xi \circ \sigma) & = & \Delta_-^\sigma \circ K^{-1} \circ \Delta_+^{\sigma*}(\xi, \xi \circ \sigma) \\ & = & \Delta_-^\sigma \circ K^{-1}(\xi). \end{array}$$

Since σ is a Lie algebra morphism, we have $[\mathfrak{g}_-^\sigma, \mathfrak{g}_-^\sigma] \subset \mathfrak{g}_+^\sigma$. This proves that $F^\sigma : \bigwedge^2 \mathfrak{g}_+^{\sigma\,*} \longrightarrow \mathfrak{g}_-^\sigma$, given by

$$F^\sigma(\xi, \eta) = p_{\mathfrak{g}_-^\sigma}[j(\xi), j(\eta)],$$

vanishes.

I will now compute φ^σ. It is defined as

$$\begin{array}{rcl} \varphi^\sigma((\xi, \sigma \circ \xi), (\eta, \sigma \circ \eta), (\nu, \sigma \circ \nu)) & = & (\nu, \sigma \circ \nu) \circ p_{\mathfrak{g}_+^\sigma}([j(\xi, \sigma \circ \xi), j(\eta, \sigma \circ \eta)]) \\ & = & \langle j(\nu, \sigma \circ \nu), [j(\xi, \sigma \circ \xi), j(\eta, \eta \circ \eta)]\rangle \\ & = & \langle \Delta_-^\sigma \circ K^{-1}(\nu), [\Delta_-^\sigma \circ K^{-1}(\xi), \Delta_-^\sigma \circ K^{-1}(\eta)\rangle \\ & = & 2K(K^{-1}(\nu), [K^{-1}(\xi), K^{-1}(\eta)]). \end{array}$$

Finally, the r-matrix is defined as

$$\begin{array}{rcc} r_{\mathfrak{d}}^\sigma : \mathfrak{g}_+^{\sigma\,*} \oplus \mathfrak{g}_-^{\sigma\,*} & \longrightarrow & \mathfrak{g}_+^\sigma \oplus \mathfrak{g}_-^\sigma \\ ((\xi, \xi \circ \sigma), (\eta, \eta \circ \sigma)) & \longmapsto & (0, j(\xi, \xi \circ \sigma)). \end{array}$$

If (ξ, η) is in $\mathfrak{d}^* = \mathfrak{g}^* \oplus \mathfrak{g}^*$ then its decomposition in $\mathfrak{g}_+^{\sigma\,*} \oplus \mathfrak{g}_-^{\sigma\,*}$ is $\frac{1}{2}((\xi + \eta \circ \sigma, \xi \circ \sigma + \eta), (\xi - \eta \circ \sigma, -\xi \circ \sigma + \eta))$. The result follows. □

I now wish to compute the bivector P_D^σ on D. By definition, it is equal to $(r_{\mathfrak{d}}^\sigma)^\lambda - (r_{\mathfrak{d}}^\sigma)^\rho$, where the upper script λ means the left invariant section of $\Gamma(TD \otimes TD)$ generated by $r_{\mathfrak{d}}^\sigma$, while the upper script ρ means the right invariant section of $\Gamma(TD \otimes TD)$ generated by $r_{\mathfrak{d}}^\sigma$.

Proposition 3.2. *Identify* T_dD *to* $\mathfrak{d}$ *by right translations. The value of* P_D^σ *at* $d = (a, b)$ *is*

$$\begin{array}{rcc} \mathfrak{d}^* = \mathfrak{g}^* \oplus \mathfrak{g}^* & \longrightarrow & \mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g} \\ (\xi, \eta) & \longmapsto & \frac{1}{2}(K^{-1}(\eta \circ \sigma \circ (\mathrm{Ad}_{\sigma(b)a^{-1}} - 1)), -K^{-1}(\xi \circ \sigma \circ (\mathrm{Ad}_{\sigma(a)b^{-1}}))). \end{array}$$

Proof. Fix $d = (a, b)$ in D. I choose to trivialise the tangent bundle, and its dual, of D by using right translations. See $(r^\sigma_{\mathfrak{d}})^\rho$ as a map from T^*D to TD. If α is in $\mathfrak{d}^*$, then

$$(r^\sigma_{\mathfrak{d}})^\rho(d)(\alpha^\rho) = (r^\sigma_{\mathfrak{d}}(\alpha))^\rho(d),$$

whereas

$$(r^\sigma_{\mathfrak{d}})^\lambda(d)(\alpha^\rho) = (\mathrm{Ad}_d \circ r^\sigma_{\mathfrak{d}}(\alpha \circ \mathrm{Ad}_d))^\rho(d).$$

Thus P^σ_D at the point $d = (a, b)$ is

$$\begin{array}{ccc} \mathfrak{d}^* = \mathfrak{g}^* \oplus \mathfrak{g}^* & \longrightarrow & \mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g} \\ (\xi, \eta) & \longmapsto & -\frac{1}{2}\Delta_- \circ K^{-1}(\xi + \eta \circ \sigma) + \frac{1}{2}\mathrm{Ad}_d \circ \Delta_- \circ K^{-1}(\xi \circ \mathrm{Ad}_a + \eta \circ \mathrm{Ad}_b \circ \sigma). \end{array}$$

The above description of P^σ_D can be simplified:

$$\begin{array}{cl} & P^\sigma_D(d)(\xi, \eta) \\ = & -\frac{1}{2}\Delta_- \circ K^{-1}(\xi + \eta \circ \sigma) + \\ & \frac{1}{2}(\mathrm{Ad}_a \circ K^{-1}(\xi \circ \mathrm{Ad}_a + \eta \circ \mathrm{Ad}_b \sigma), -\sigma \circ \mathrm{Ad}_{\sigma(b)} \circ K^{-1}(\xi \circ \mathrm{Ad}_a + \eta \circ \mathrm{Ad}_b \circ \sigma)) \\ = & -\frac{1}{2}\Delta_- \circ K^{-1}(\xi + \eta \circ \sigma) + \\ & \frac{1}{2}(K^{-1}(\xi + \eta \circ \mathrm{Ad}_b \sigma \circ \mathrm{Ad}_{a^{-1}}), -\sigma \circ K^{-1}(\xi \circ \mathrm{Ad}_{a\sigma(b)^{-1}} + \eta \circ \mathrm{Ad}_b \circ \sigma \circ \mathrm{Ad}_{\sigma(b)^{-1}})) \\ = & -\frac{1}{2}(K^{-1}(\xi + \eta \circ \sigma), -\sigma \circ K^{-1}(\xi + \eta \circ \sigma)) + \\ & \frac{1}{2}(K^{-1}(\xi + \eta \circ \sigma \circ \mathrm{Ad}_{\sigma(b)a^{-1}}), -\sigma \circ K^{-1}(\xi \circ \mathrm{Ad}_{a\sigma(b)^{-1}} + \eta \circ \sigma)) \\ = & \frac{1}{2}(K^{-1}(\eta \circ \sigma \circ (\mathrm{Ad}_{\sigma(b)a^{-1}} - 1)), -\sigma \circ K^{-1}(\xi \circ (\mathrm{Ad}_{a\sigma(b)^{-1}} - 1))) \\ = & \frac{1}{2}(K^{-1}(\eta \circ \sigma \circ (\mathrm{Ad}_{\sigma(b)a^{-1}} - 1)), -K^{-1}(\xi \circ \sigma \circ (\mathrm{Ad}_{\sigma(a)b^{-1}} - 1))). \end{array}$$

□

It follows from [1] that P^σ_D is projectable on $S^\sigma = D/G^\sigma_+$. Actually, the following is also true

Proposition 3.3. *The bivector P^σ_D is projectable to a bivector P^σ_S on $S = D/G_+$. Identify S with G through the map*

$$\begin{array}{ccc} D & \longrightarrow & G \\ (a, b) & \longmapsto & ab^{-1}. \end{array}$$

Trivialise the tangent space to G, and hence to S, by right translations. If s is in S, then using the above identification, P^σ_S at the point s is

$$P^\sigma_S(s)(\xi) = \frac{1}{2}(\mathrm{Ad}_{\sigma(s)^{-1}} - \mathrm{Ad}_s) \circ \sigma \circ K^{-1}(\xi). \tag{2}$$

Proof. Assume s in S is the image of (a, b) in D, that is $s = ab^{-1}$. The tangent map of

$$\begin{array}{ccc} D & \longrightarrow & G \\ (a, b) & \longmapsto & ab^{-1} \end{array}$$

at (a, b) is

$$\begin{array}{ccc} p : \mathfrak{d} & \longrightarrow & \mathfrak{g} \\ (x, y) & \longmapsto & x - \mathrm{Ad}_{ab^{-1}} y. \end{array}$$

The dual map of p is

$$\begin{array}{ccc} p^* : \mathfrak{g}^* & \longrightarrow & \mathfrak{d}^* \\ \xi & \longmapsto & (\xi, -\xi \circ \mathrm{Ad}_{ab^{-1}}) \end{array}$$

The bivector P^σ_D is projectable onto S if and only if for all (a, b) in D and ξ in $\mathfrak{g}^*$, the expression

$$p(P^\sigma_D(a, b)(p^*\xi))$$

depends only on $s = ab^{-1}$. It will then be equal to $P_S^\sigma(s)(\xi)$. This expression is equal to

$$\begin{aligned} & p(P_D^\sigma(a,b)(\xi, -\xi \circ \mathrm{Ad}_{ab^{-1}})) \\ = \quad & \tfrac{1}{2} p((\mathrm{Ad}_{a\sigma(b)^{-1}} - 1) \circ \sigma \circ K^{-1}(-\xi \circ \mathrm{Ad}_{ab^{-1}}), (1 - \mathrm{Ad}_{b\sigma(a)^{-1}}) \circ \sigma \circ K^{-1}(\xi)) \\ = \quad & \tfrac{1}{2} p((\mathrm{Ad}_{\sigma(ba^{-1})} - \mathrm{Ad}_{a\sigma(a)^{-1}}) \circ \sigma \circ K^{-1}(\xi), (1 - \mathrm{Ad}_{b\sigma(a)^{-1}}) \circ \sigma \circ K^{-1}(\xi)) \\ = \quad & \tfrac{1}{2} (\mathrm{Ad}_{\sigma(ba^{-1})} - \mathrm{Ad}_{a\sigma(a)^{-1}} - \mathrm{Ad}_{ab^{-1}} + \mathrm{Ad}_{a\sigma(a)^{-1}}) \circ \sigma \circ K^{-1}(\xi) \\ = \quad & \tfrac{1}{2} (\mathrm{Ad}_{\sigma(ba^{-1})} - \mathrm{Ad}_{ab^{-1}}) \circ \sigma \circ K^{-1}(\xi). \end{aligned}$$

This both proves that P_D^σ is projectable on S and gives a formula for the projected bivector. □

To prove that there exists a quasi-Poisson action of G^σ on (S, P_S^σ), I must compute $[P_S^\sigma, P_S^\sigma]$, where $[,]$ is the Schouten-Nijenhuis bracket on multi-vector fields.

Lemma 3.4. *For x, y and z in $\mathfrak{g}$, let $\xi = K(x)$, $\eta = K(y)$ and $\nu = K(z)$. We have*

$$\frac{1}{2}[P_S^\sigma(s), P_S^\sigma(s)](\xi, \eta, \nu) = \frac{1}{4} K(x, [y, \tau_s(z)] + [\tau_s(y), z] - \tau_s([y, z])),$$

where $\tau_s = \mathrm{Ad}_s \circ \sigma - \sigma \circ \mathrm{Ad}_{s^{-1}}$.

Proof. Let (a, b) in D be such that $s = ab^{-1}$. Let p be as in the proof of Proposition 3.3. The bivector $P_S^\sigma(s)$ is $p(P_D^\sigma(a, b))$. Hence,

$$[P_S^\sigma(s), P_S^\sigma(s)] = p([P_D^\sigma(a, b), P_D^\sigma(a, b)]).$$

But it is proved in [1] that

$$[P_D^\sigma(a, b), P_D^\sigma(a, b)] = (\varphi^\sigma)^\rho(a, b) - (\varphi^\sigma)^\lambda(a, b).$$

Hence

$$\frac{1}{2}[P_S^\sigma(s), P_S^\sigma(s)] = p((\varphi^\sigma)^\rho(a, b)) - p((\varphi^\sigma)^\lambda(a, b)).$$

Now, it is tedious but straightforward and very similar to the above computations to check that

$$p((\varphi^\sigma)^\rho(a, b))(\xi, \eta, \nu) = \frac{1}{4} K(x, [y, \tau_s(z)] + [\tau_s(y), z] - \tau_s([y, z])),$$

and

$$p((\varphi^\sigma)^\lambda(a, b))(\xi, \eta, \nu) = 0.$$

□

The group D acts on $S = D/G_+$ by multiplication on the left. This action restricts to an action of G_+^σ on S. Identifying G and G_+^σ via Δ_+^σ, this action is

$$\begin{array}{ccc} G \times S & \longrightarrow & S \\ (g, s) & \longmapsto & gs\sigma(g)^{-1}. \end{array}$$

The infinitesimal action of $\mathfrak{g}$ at the point s in S reads

$$\begin{array}{ccc} \mathfrak{g} & \longrightarrow & T_s S \simeq \mathfrak{g} \\ x & \longmapsto & x - \mathrm{Ad}_s \circ \sigma(x), \end{array}$$

with dual map

$$\begin{array}{ccc} T_s^* S \simeq \mathfrak{g}^* & \longrightarrow & \mathfrak{g}^* \\ \xi & \longmapsto & \xi - \xi \circ \mathrm{Ad}_s \circ \sigma. \end{array}$$

Denote by $(\varphi^\sigma)_S$ the induced trivector field on S. If ξ, η and ν are in $\mathfrak{g}^*$ then

$$(\varphi^\sigma)_S(s)(\xi, \eta, \nu) = \varphi^\sigma(\xi - \xi \circ \mathrm{Ad}_s \circ \sigma, \eta - \eta \circ \mathrm{Ad}_s \circ \sigma, \nu - \nu \circ \mathrm{Ad}_s \circ \sigma).$$

Computing the right hand side in the above equality is a simple calculation which proves the following Lemma.

Lemma 3.5. *The bivector field P^σ_S and the trivector field $(\varphi^\sigma)_S$ satisfy*

$$\frac{1}{2}[P^\sigma_S, P^\sigma_S] = (\varphi^\sigma)_S.$$

To prove that the action of G^σ_+ on (S, P^σ_S) is indeed quasi-Poisson, there only remains to prove that P^σ_S is G^σ_+-invariant.

Lemma 3.6. *The bivector field P^σ_S is G^σ_+-invariant.*

Proof. Fix g in $G \simeq G^\sigma_+$. Denote Σ_g the action of g on S. The tangent map of Σ_g at $s \in S$ is

$$\begin{array}{ccc} T_sS \simeq \mathfrak{g} & \longrightarrow & T_{gs\sigma(g)^{-1}}S \simeq \mathfrak{g} \\ x & \longmapsto & \mathrm{Ad}_g x. \end{array}$$

Also, if ξ is in $\mathfrak{g}^*$

$$\begin{array}{cl} & P^\sigma_S(gs\sigma(g)^{-1})(\xi) \\ = & \frac{1}{2}(\mathrm{Ad}_{g\sigma(s)^{-1}\sigma(g)^{-1}} - \mathrm{Ad}_{gs\sigma(g)^{-1}}) \circ \sigma \circ K^{-1}(\xi) \\ = & \frac{1}{2}\mathrm{Ad}_g \circ (\mathrm{Ad}_{\sigma(s)^{-1}} - \mathrm{Ad}_s) \circ \mathrm{Ad}_{\sigma(g)^{-1}} \circ \sigma \circ K^{-1}(\xi) \\ = & \mathrm{Ad}_g(P^\sigma_S(s)(\xi \circ \mathrm{Ad}_g)) \\ = & (\Sigma_g)_*(P^\sigma_S)(\Sigma_g(s))(\xi). \end{array}$$

□

Lemma 3.7. *Let s be in S. The image of $P^\sigma_S(s)$ is*

$$\mathrm{Im}P^\sigma_S(s) = \{(1 - \mathrm{Ad}_s \circ \sigma) \circ (1 + \mathrm{Ad}_s \circ \sigma)(y) \mid y \in \mathfrak{g}\}.$$

In particular, it is included in the tangent space to the orbit through s of the action of G^σ.

Proof. The image of $P^\sigma_S(s)$ is by Proposition 3.3

$$\mathrm{Im}P^\sigma_S(s) = \{(\mathrm{Ad}_{\sigma(s)^{-1}} - \mathrm{Ad}_s)\sigma(x) \mid x \in \mathfrak{g}\}.$$

The Lemma follows by setting $x = \mathrm{Ad}_s \circ \sigma(y) = \sigma \circ \mathrm{Ad}_{\sigma(s)}(y)$ and noticing that $(1 - (\mathrm{Ad}_s \circ \sigma)^2) = (1 - \mathrm{Ad}_s \circ \sigma) \circ (1 + \mathrm{Ad}_s \circ \sigma)$. □

This finishes the proof of Theorem 2.1.

Choose G and σ as in Theorem 2.2. The trivector field $[P^\sigma_S, P^\sigma_S]$ is tangent to the orbit of the action of G^σ_+ on S. These orbits are of dimension at most 2, therefore the trivector field $[P^\sigma_S, P^\sigma_S]$ vanishes and P^σ_S defines a Poisson structure on $\mathrm{SL}(2,\mathbb{R})$ which is invariant under the action

$$\begin{array}{ccc} \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) & \longrightarrow & \mathrm{SL}(2,\mathbb{R}) \\ (g, s) & \longmapsto & gs\sigma(g)^{-1}. \end{array}$$

Lemma 3.7 and a simple computation prove that along the orbit of the identity, the bivector field P^σ_S vanishes; and that elsewhere, its image coincides with the tangent space to the orbits of the above action. Recall that in [4], the domain **I** is given by

$$z(\tau,\theta,\rho) = \begin{bmatrix} \sinh(\frac{\rho}{2}) + \cosh(\frac{\rho}{2})\cos(\tau) & \exp(\theta)\cosh(\frac{\rho}{2})\sin(\tau) \\ -\exp(-\theta)\cosh(\frac{\rho}{2})\sin(\tau) & -\sinh(\frac{\rho}{2}) + \cosh(\frac{\rho}{2})\cos(\tau) \end{bmatrix}. \tag{3}$$

This formula also defines coordinates on I. Using Formula (2) of Proposition 3.3 and a computer, it is easy to check that P^σ_S if indeed given by Formula (1). This ends the proof of Theorem 2.2.

4. A final remark

One might ask how different is the quasi-Poisson action of G_+^σ on (S, P_S^σ) from the usual quasi-Poisson action of G_+ on (S, P_S). For example, if one takes $G = \mathrm{SU}(2)$, $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\sigma = \mathrm{Ad}_H$ then the multiplication on the right in $\mathrm{SU}(2)$ by $\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ defines an isomorphism between the two quasi-Poisson actions.

Nevertheless, in the example of Theorem 2.2, the two structures are indeed different since for example the action of $\mathrm{SL}(2, \mathbb{R})$ on itself by conjugation has two fixed points whereas the action of $\mathrm{SL}(2, \mathbb{R})$ on itself used in Theorem 2.2 does not have any fixed point.

References

[1] A. Alekseev and Y. Kosmann-Schwarzbach – *Manin pairs and moment maps*, J. Differential Geom. 56 (2000), no. 1, 133–165.

[2] A. Alekseev, Y. Kosmann-Schwarzbach and E. Meinrenken – *Quasi-Poisson manifolds*, Canad. J. Math. 54 (2002), no. 1, 3–29.

[3] P. Bieliavsky, S. Detournay, Ph. Spindel and M. Rooman – *Star Product on extended massive non-rotating BTZ black holes*, J. High Energy Phys. JHEP06(2004)031.

[4] P. Bieliavsky, M. Rooman and Ph. Spindel – *Regular Poisson structures on massive non-rotating BTZ black holes*, Nuclear Phys. B 645 (2002), no. 1-2, 349–364.

Department of Mathematics, South Kensington Campus, Imperial College London, SW7 2AZ, UK

E-mail address: `s.racaniere@ic.ac.uk`

URL: `www.ma.ic.ac.uk/~racani`